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Stochastic Model for Dielectric Breakdown

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We discuss a model for the development of discharge patterns in dielectric breakdown based on the Laplace equation associated with a probability field. The model gives rise to random fractals with well-defined Hausdorff dimensions. The relations of this model with the diffusion-limited aggregation are discussed in detail. The possibility of application to other stochastic phenomena like fracture propagation is proposed.

KEY WORDS: Stochastic growth models; fractal structures; dielectric breakdown.

1. INTRODUCTION

The question we would like to discuss is why a lightning bolt does not proceed in a straight line or, more generally, whether it is possible to identify the mechanisms that give rise to its complicated pattern. The behavior of atmospheric lightning is of course complicated by the particular geometry, the nonhomogeneity of the atmosphere, and the local dielectric properties of the air. In order to avoid these complications it is more convenient to look at the abundant amount of experimental data on dielectric breakdown in gases, liquids, and solids.[1–3]

The phenomenon of dielectric breakdown frequently occurs by means of narrow discharge channels that exhibit a strong tendency to branching into complicated stochastic patterns. The global structures of branched discharges often show a close structural similarity within a large class of discharge types but at the moment even a qualitative classification of these structures is missing. On one hand there is a tendency for the discharge to grow on the points where the electric field is highest. Because of electrostatic reasons

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these points are the tips of the branches. On the other hand, if this effect would be dominating, the breakdown would just proceed along a straight line toward the other electrode. We observe instead a strong tendency to ramify, an indication that the process has a stochastic nature instead of a deterministic one. It is therefore the competition between electrostatic and stochastic effect that governs the phenomenon.

We have recently proposed a model[4] that provides, in our opinion, the simplest nontrivial description of these competing effects. The model is based on the idea that the local electric field around a discharge pattern does not govern the growth directly but through a stochastic process. This means that the pattern does not just grow at the point of maximum local field but that at this point the probability of growing is the highest. This changes the process from deterministic into stochastic and gives rise to the possibility of branching in a statistical sense. The physical reason for this probabilistic description of the phenomenon lies in the following fact: the electric field as it is calculated in the model is determined by the global structure of the discharge pattern. It reflects the quasistatic influence of the pattern on the local conditions for its growth. But it does not contain the fluctuations of the field and charge densities as they occur in the real dynamical process, especially at the tips of the filaments. These fluctuations are the origin of the stochastic nature of the process and make branching possible. The relation between probability and field reflects therefore the fact that the microscopic mechanism of the propagation of the discharge is modulated by its global structure.

It is interesting to note that there are other phenomena that also give rise to ramified structures and for which a similar approach may be appropriate. This is for example the case for the propagation of fractures in a solid medium.[5] Fractures tend to proceed along the points of maximum strain but, if an area of weak resistance gets close to a point of medium strain, the propagation will rather proceed there in close analogy to the dielectric breakdown. The fractures of a broken glass are in fact not too dissimilar in their global structure from the shape of electrical discharges. We are going to see in the following that the mathematical analogy is actually very close.

In Section 2 we analyze an experiment designed to produce a two-dimensional discharge pattern in a gas. Evidence for a fractal structure is provided. In Section 3 our stochastic model is described and the resulting fractal patterns are briefly discussed. In Section 4 the relation of our model to the model of diffusion-limited aggregation is discussed in detail.

2. ANALYSIS OF EXPERIMENTAL DISCHARGES

The highly branched patterns observed in dielectric breakdown suggest the possibility of fractal structures.[6] Their analysis is often complicated by the fact that photos represent two-dimensional projections of three-dimensional patterns and that often they refer to a superposition of discharge patterns occurring at different times. In order to facilitate the analysis of the discharge patterns the optimal situation would be that of a two-dimensional radial discharge. This has been recently obtained by Niemeyer and Pinnekamp for a leader surface discharge (Lichtenberg figure) in compressed SF_6 gas.[7] The experimental parameters were controlled in such a way that (i) the voltage drop along the discharge channels is small compared to the applied voltage, so that the channels form approximately an equipotential structure; (ii) the electric field exhibits cylindrical symmetry and has its outer boundary at large distance from the discharge pattern; (iii) the growth of a discharge pattern occurs during a well-defined time interval (of order 1 μsec) during which the channels remain conductive and preserve their equipotential behavior. The experiment thus, to a good approximation, produces an equipotential channel system growing in a plane with radial symmetry from a central point. An example of these figures was reported in Fig. 1 of Ref. 4. It should be noted anyhow that the thickness of the branches in the photo does not correspond to a real thickness but is due to the amount of charge that has flown through that channel. In addition several tiny branches disappear in the printed reproduction. To avoid these problems and give a realistic view of the discharge pattern we have drawn all the lines appearing in the original negative with equal thickness. The resulting picture is shown in Fig. 1a.

A fractal object is characterized by a noninteger power law between "mass" (N) and radius (r):

$$N \sim r^D \tag{2.1}$$

where D is the Hausdorff or fractal dimension.[6] If the system is composed by filamentary branches as in our case the number of branches $n(r)$ at a distance r is given by the gradient of the mass:

$$n(r) \sim \frac{dN(r)}{dr} \sim r^{(D-1)} \tag{2.2}$$

Therefore a simple count of the number of branches at various distances provides a measure of $(D-1)$. The analysis of Fig. 1 indicates a power law with $D \sim 1.7$. The indetermination is rather large owing to the small size of the system.

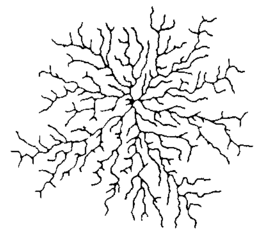
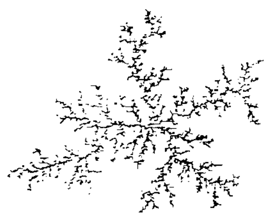
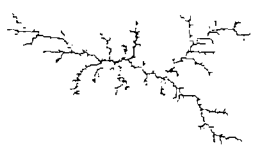
Planar discharge in SF_6 gas. This experiment corresponds to an equipotential channel system growing in a plane with radial electrode . For this structure we estimate $D \sim 1.7$	
Stochastic model with $\eta = 0.5$ Hausdorff dimension : $D = 1.89 \pm 0.01$	
Stochastic model with $\eta = 1.0$ Hausdorff dimension : $D = 1.75 \pm 0.02$ (in this case the growth probability is proportional to the local field)	
Stochastic model with $\eta = 2.0$ Hausdorff dimension : $D \cong 1.6$	

Fig. 1. (a) Channel pattern extracted from the photo of Ref. 4. (b, c, d) Structures arising from the stochastic model for different values of η.

3. STOCHASTIC MODEL

We briefly summarize here the assumptions and the results of the stochastic model we have introduced in Ref. 4. The model is defined on a lattice and it is based on the following simple rules:

(a) The discharge pattern is *equipotential* and connected.

(b) The probability p_j to add the segment j connected to the pattern is related to the *local electric field* E_j.

As for the relation between p_j and E_j we consider a power law dependence characterized by the exponent η:

$$p_j \propto E_j^\eta \tag{3.1}$$

The particular relation between probability and local field should be linked to microscopic models of dielectric breakdown. (The case $\eta = 1$ corresponds to simple proportionality between probability and field.) For more details about the model see Ref. 4.

The above rules imply the solution at each step of the Laplace equation with the appropriate boundary conditions defined by the growing equipotential pattern. We assume $\phi = 0$ along the discharge pattern and $\phi = 1$ outside a circle at large enough distance.[4] The local electric field (potential drop) along a segment that links a point of the discharge pattern to a nearest-neighbor point that is not yet part of it but which is a candidate for the growth, is given by the value of ϕ at this point. The probability field for the addition of a segment is therefore related to the values of the potential in all the points surrounding the already available discharge pattern.[4] To determine it, the discrete Laplace equation

$$4\phi_{i,k} - (\phi_{i+1,k} + \phi_{i-1,k} + \phi_{i,k+1} + \phi_{i,k-1}) = 0 \tag{3.2}$$

is solved by iterations with the above-specified boundary conditions. These rules properly define the starting of the process from the central point and it also follows that no crossing is possible and that the pattern is simply connected.

Some examples of the structures generated by computer studies of this stochastic model in a plane are shown in Fig. 1b, c, d. They are self-similar random fractals with a well-defined Hausdorff dimension D that explicitly depends on the exponent η used in Eq. (3.1). The dimensionality is defined by the slope in a log–log plot, of the total number of bonds (belonging to the pattern) within a certain radius as a function of the radius itself. The limit $\eta = 0$ gives $D = 2$ in analogy with the Eden model.[8] The limit of large η is difficult to study numerically because the structure has little ramification and a huge starting grid would be necessary for a good statistics.

The above model can be extended in a natural way to describe other types of stochastic growth phenomena such as, for example, the propagation of fracture lines in a solid.[5] In this case the equilibrium equation for the elastic body is different from the Laplace equation but it is still of elliptic type. The nature of the boundary value problem that defines the probability field for further growth is therefore rather similar. In this case the stochastic nature of the process is related to the distribution of points of weak resistance in the material.

4. RELATION TO THE MODEL OF DIFFUSION-LIMITED AGGREGATION

In this section we discuss the relations of the present model with the diffusion-limited aggregation (DLA). In particular we will show that in the continuum limit our model with $\eta = 1$ and the DLA model coincide. No simple relation can be found instead for the case where $\eta \neq 1$.

The DLA model was introduced in 1981 by Witten and Sander[9] and it has been object of extensive studies since then.[10,11] The model simulates random growth by starting a seed particle at the origin of a lattice. Another particle is allowed to random walk from a distant lattice point until it reaches a site that is nearest to the seed lattice site; it is then halted and another one is started from far away and allowed to walk until it can attach to the previous two particles and so on. This model is a natural starting point for a description of nonequilibrium aggregation.

The dynamics of a random walk on a lattice can be described by a master equation

$$\Gamma^{-1} \frac{d\rho_{i,k}(t)}{dt} = -4\rho_{i,k}(t) + \sum_{\langle n,n \rangle} \rho_{i',k'}(t) \tag{4.1}$$

Here for simplicity we have considered a two-dimensional square lattice but the extension to other cases is trivial. In equation (4.1) $\rho_{i,k}(t)$ represents the probability (or the average occupation density for the case of many particles) to find the particle at the site with coordinates (i, k) at time t. The jump rate is indicated by Γ and the sum runs over the nearest neighbors of the site (i, k).

It is clear that the right side of Eq. (4.1) is identical to the discretized Laplace operator applied to ρ. The stationary solutions $\rho^{\rm st}_{i,k}$ will correspond therefore to the potential ϕ of (3.2):

$$\frac{d\rho^{\rm st}_{i,k}}{dt} = 0 \tag{4.2}$$

$$-4\rho^{\rm st}_{i,k} + \sum_{\langle n,n \rangle} \rho^{\rm st}_{i',k'} = 0 \tag{4.3}$$

The boundary conditions of the electric problem can be interpreted in the following sense: $\phi = 1$ on a far away circle corresponds to a continuous source of particles on this circle with $\rho^{st}_{i,k} = 1$; $\phi = 0$ on the growing structure implies instead that this structure is a sink for the diffusing particles with $\rho^{st}_{i,k} = 0$. The field $\phi_{i,k}$ at a given point of the lattice corresponds to the average occupation of this point in a stationary diffusion process with source and sink defined by the above boundary value problem. The condition $d\rho/dt = 0$ is therefore exact for DLA and not just an approximation valid in certain limits as discussed by Witten and Sander.[11] (Our ρ here is u of Ref. 11). In the case of the DLA model the probability field for the stationary diffusion process (average occupation of the sites nearest to the growing pattern) is not computed explicitly but it is directly probed by the random walk of a single particle. In our case we compute the potential field (or occupation density) explicitly and then we construct a probability field that is not necessarily identical to it ($\eta \neq 1$). The final probability field is then probed by a random number generator.

It is clear therefore that there is a one-to-one correspondence between our model with $\eta = 1$ and the DLA model, although the interpretation is different: bond model for dielectric breakdown against site model for DLA. An important remark should be made at this point concerning how the DLA model is actually realized in the computer simulations. The diffusing particle is stopped when it reaches a site that is nearest to the available pattern. This corresponds to our model only in the continuum limit. In fact to have a proper correspondence also for the discrete case one should continue the random walk until it is actually absorbed by the pattern and then add to the pattern the last step of this walk. This would be the correct way to probe the probability field described by Eq. (4.3). The difference between the two processes is, of course, expected to be negligible when the size of the object becomes substantially larger than the lattice spacing.

A question one may consider is then whether our cases with $\eta \neq 1$ have an analogy in terms of random walk models. Since in DLA the probability field is never constructed explicitly this relation is not easy to see. All one can do in the DLA is to change the dimensionality of the walk and use for example walks with different dimension like the self-avoiding walk or others instead of a simple random walk with dimension $D_w = 2$. This would completely change the master equation (4.1) and it is hard to see at the moment any relation between the dimension of the walk used in a DLA process and our parameter η.

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